the former case, it is found that F_2 and its first two derivatives contain errors of order, at most, 6% over the range of ζ of practical interest. Up to $\zeta = 1.4$, $F_2''(\zeta)$ underestimates the value (by less than 5% at $\zeta = 0$) and then overestimates, whereas F_2' and F_2 underestimate for all ζ.

In the latter case, H_1 and F_2 have errors of the order of 30% which get much bigger for certain ranges of ζ. The functions H_2 and F_3 were calculated for this case, and they led to a noticeable improvement in accuracy, the errors then in H_2 and H_2' being roughly 20%, whereas those in the F_3 functions were 10–15%. More important, perhaps, bearing in mind the "pincer-movement" detected by Weyl for the $\beta = 0$ case when establishing convergence, we find the same phenomenon appearing to begin here, viz., $F_2 > F > F_3$ and $H_1 < H < H_2$ for all ζ.

Finally, in connection with the possible use of the method in predicting rough starting values for use in standard step-by-step methods of numerical integration we note that the error in $H_1'(0)$ is 39% and in $H_2'(0)$ is 26%, whereas in $F_2''(0)$ it is 32% and in $F_3''(0)$, 12.%

Appendix

The derivation of formula (8) for $F_2'(\zeta)$ requires the vanishing of the following expressions:

$$\left[\zeta \int_\infty^\zeta \exp\left[\frac{-cu^2}{2}\right] \int_0^u \exp\left[\frac{(1-\epsilon)ct^2}{2}\right] \times \int_0^t \exp\left[\frac{-\epsilon cs^2}{2}\right] ds\, dt\, du\right]_0^\infty \quad (9)$$

$$\left\{\exp\left[\frac{-cv^2}{2}\right] \int_0^v \exp\left[\frac{(1-\epsilon)ct^2}{2}\right] \times \int_0^t \exp\left[\frac{-\epsilon cs^2}{2}\right] ds\, dt\right\}_0^\infty \quad (10)$$

In proving that these expressions are zero, we make use of the inequality,

$$\int_0^t \exp\left[\frac{-\epsilon cs^2}{2}\right] ds \leqslant t$$

$$0 \leqslant t < \infty \qquad (\epsilon, c \geqslant 0) \quad (11)$$

Then the magnitude of the bracketed expression in (9) is

$$\leqslant \frac{\zeta}{c|1-\epsilon|} \int_\zeta^\infty \left| \exp\left[\frac{-\epsilon cu^2}{2}\right] - \exp\left[\frac{-cu^2}{2}\right] \right| du \quad (12)$$

It follows that (9) vanishes at the upper limit because

$$\zeta \int_\zeta^\infty e^{-u^2}\, du \leqslant \zeta \int_\zeta^\infty e^{-u} du = \zeta e^{-\zeta} \to 0 \text{ as } \zeta \to \infty$$

and at the lower limit because both terms in the integral in (12) can be transformed into error functions and have finite constant values at $\zeta = 0$.

Similarly, by using (11), we note that the magnitude of the bracketed expression in (10) is

$$\leqslant \frac{\exp(-\epsilon cv^2/2) - \exp(-cv^2/2)}{c(1-\epsilon)}$$

which clearly tends to zero as $v \to \infty$ for all positive ϵ. At the lower limit we observe that the integral in (10) $\sim v^2/2$ for small v which establishes the result.

The preceding argument is not valid as it stands when $\epsilon = 1$, but in that case the original expressions (9) and (10) are simplified, and minor modifications may be introduced to show that the results are still true.

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Effect of Thermal Radiation on Thin Shock Structure

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1. Introduction

In this note we shall report some findings through the application of "boundary-layer technique" on the problem of shock transition with thermal radiation. We approach the problem of studying shock transition by considering the viscosity and the heat-conduction effects of the gas in addition to the radiation effect. Then, in the limiting process, we examine the physical situation corresponding to a shock of vanishing thickness. Whenever we refer to the shock in the following we mean only the region in which viscosity and heat conduction are important. The mathematical theory of the study of the nonradiating shock transition has been very well established.[3] The boundary-layer nature of the shock transition allows one to study the flow field of the shock structure as well as the flow field outside the shock in a systematic manner based on a small parameter $1/Re$, where Re is the reference Reynolds number.[4] This technique has been used in the present analysis of the study of the shock transition with thermal-radiation effect. It will be seen that in this case, a new parameter $\nu = 0(\bar{\alpha}/Re)$ will play an important role in the analysis, where $\bar{\alpha}$ is the reference value of the absorption coefficient of the gas medium. In the limit as ν approaches zero, the flow field will correspond to the case of an infinitesimally thin shock. It will also be seen that the shock structure of the lowest order is modified due to the presence of the thermal-radiation effect. Included in Ref. 5 and 8 are some altogether differenta pproaches to the radiative shock-structure problem.

2. Governing Flow Equations

The basic equations for the steady one-dimensional flow can be expressed in the following form:

$$d/dx\ (\rho u) = 0 \quad (1a)$$

$$(\rho u)\ du/dx + dp/dx - d\tau/dx = 0 \quad (1b)$$

$$(\rho u)\ d/dx(h + \tfrac{1}{2}u^2) + d/dx(q + q_r - u\tau) = 0 \quad (1c)$$

$$p = [(\gamma - 1)/\gamma]\,\rho h \quad \text{(perfect gas)}$$

These equations can be integrated once to give

$$\rho u = \Gamma \quad (2a)$$

$$\Gamma u + [(\gamma - 1)/\gamma]\Gamma h/u - \tau = \Gamma c_1 \quad (2b)$$

$$\Gamma(h + \tfrac{1}{2}u^2) + (q + q_r - u\tau) = \Gamma c_2 \quad (2c)$$

The shearing stress τ and the heat flux due to heat conduction q can be expressed as $\tau = \mu/L\ (du/dx)$ and $q = -k/Lc_p\ (dh/$

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dx). For the radiative heat flux q_r, we use the model devised by Vincenti and Baldwin[6] for a nonscattering gray gas. Using a sign convention consistent with the independent variable x allows q_r to be expressed as

$$q_r(\xi) = \frac{2m\sigma}{nc_p^4}\int_{-\infty}^{+\infty} h^4(\xi')\,\mathrm{sgn}(\xi - \xi')\exp(-|\xi - \xi'|)\,d\xi'$$

$$\xi = n\int_0^x L\alpha(x')dx'$$

Assuming Prandtl number is equal to unity, Eqs. (2) can be transformed into the form

$$v + \left(\frac{\gamma - 1}{\gamma}\right)\frac{\bar{h}}{v} = 1 + \nu^*\frac{dv}{d\xi} \quad (3a)$$

$$(\bar{h} + \tfrac{1}{2}v^2) = C + \nu^*\, d/d\xi\,(\bar{h} + \tfrac{1}{2}v^2) - QF \quad (3b)$$

with the following mathematical definitions:

$$\nu^*(x) = \frac{\mu(x)}{\bar{\mu}} \times \frac{\alpha(x)}{\bar{\alpha}} \times \frac{n\bar{\alpha}\bar{\mu}}{\Gamma} = \left(\frac{\mu}{\bar{\mu}}\right)\left(\frac{\alpha}{\bar{\alpha}}\right)\left(\frac{n\bar{\alpha}}{Re}\right) = \left[\frac{\mu}{\bar{\mu}}\cdot\frac{\alpha}{\bar{\alpha}}\right]\nu \qquad \bar{\alpha}\times 1 = \int_0^1 \alpha(x)dx$$

$$C = c_2/c_1^2 \qquad u = c_1 v \qquad h = c_1^2\bar{h}$$

$$Q = (2n/3)(\sigma c_1^6/\Gamma c_p^4) \qquad \bar{\mu} = \text{const}$$

$$F = \int_{-\infty}^{+\infty} \bar{h}^4(\xi')\,\mathrm{sgn}(\xi - \xi')\exp(-|\xi - \xi'|)\,d\xi'$$

Equations (3a) and (3b) are the governing equations for the dependent variables v and $\bar{h}$. Equation (3a) is the same as the momentum equation in the usual shock-transition problem. The energy equation, Eq. (3b), has an extra term QF due to the presence of the thermal-radiation effect. In the next section we explain the proposed scheme of solving these equations by an expansion technique based on the small parameter ν. In the following analysis, we have assumed $\alpha = \bar{\alpha}$ and $\mu = \bar{\mu}$. These assumptions will not affect the generality or the qualitative nature of our conclusions.

3. Expansion Scheme and the Separation of the Radiation-Flux Integral

We propose to divide the flow field into regions I, II, and III. The coordinate system is chosen in such a way that the shock (which has a thickness of the order of ν) is included within region II. Regions I and III refer to flow fields upstream and downstream of the shock, respectively.

In region II, the viscosity and the heat-conduction effects are important because the gradients of v and $\bar{h}$ are large, whereas in regions I and III, these effects are of higher order. It has been established[3] that, for this type of problem, if we stretch the independent variable ξ in region II by multiplying it by a factor of $(1/\nu)$, then we can consider the v, $\bar{h}$ functions and their derivatives with respect to the new independent variables to have the same order of magnitude in all three regions. This suggests the following expansion scheme for the flow variables v and $\bar{h}$ in the three regions in terms of the small parameter ν, regions I and III:

$$_i\bar{h}(\xi) = {}_1\bar{h}_0(\xi) + \nu_1\bar{h}_1(\xi) + o(\nu) \quad (4a)$$

$$_iv(\xi) = {}_1v_0(\xi) + \nu_1v_1(\xi) + o(\nu) \quad (4b)$$

where subscript $i = 1, 3$ refers to regions I and III, respectively.

region II (the region containing the shock)

$$\bar{h}(y) = \bar{h}_0(y) + \nu\bar{h}_1(y) + o(\nu) \quad (5a)$$

$$v(y) = v_0(y) + \nu v_1(y) + o(\nu) \qquad y = \xi/\nu \quad (5b)$$

Equations (4) and (5) are ready to be introduced into Eqs. (3) to obtain the successive orders of governing equations for the variables $\bar{h}_0(y)$, $v_0(y)$, $\bar{h}_1(y)$, etc. But before doing so, the radiation-flux integral should be expressed in integral forms suitable for expansion. For instance, for $-\nu \leq \xi, y \leq +\nu$

$$F = F_2 = \int_{-\infty}^{0^-} {}_1\bar{h}^4(\xi')e^{-|\xi-\xi'|}\,d\xi' + \nu\int_{-\infty} \bar{h}^4(y')e^{-\nu|y-y'|}\,dy' - \nu\int_y^{+\infty}\bar{h}^4(y')e^{-\nu|y-y'|}\,dy' - \int_{0^+}^{+\infty} {}_3\bar{h}^4(\xi')e^{-|\xi-\xi'|}\,d\xi' \quad (6)$$

4. Thin Shock-Structure Equations and Corresponding Shock Conditions

The governing equations of successive orders in the three regions can be obtained by collecting terms of like power of the small parameter ν. Those of the lowest order have been obtained by taking the limit of ν approaching zero.

Region II:

$$v_0 + \left(\frac{\gamma - 1}{\gamma}\right)\frac{\bar{h}_0}{v_0} = 1 + \frac{dv_0}{dy} \quad (7a)$$

$$\left(\bar{h}_0 + \frac{1}{2}v_0^2\right) = C + \frac{d}{dy}\left(\bar{h}_0 + \frac{1}{2}v_0^2\right) - Q\left\{\int_{-\infty}^{0^-} {}_1\bar{h}_0^4(\xi')e^{-|\xi'|} \times d\xi' - \int_{0^+}^{+\infty} {}_3\bar{h}_0^4(\xi')e^{-|\xi'|}\,d\xi'\right\} \quad (7b)$$

Regions I and III:

$${}_1v_0 + [(\gamma - 1)/\gamma]_1\bar{h}_0/{}_1v_0 = 1 \quad (8a)$$

$$\left({}_1\bar{h}_0 + \frac{1}{2}{}_1v_0^2\right) = C - Q\left\{\int_{-\infty}^{\xi} {}_1\bar{h}_0^4(\xi')e^{-|\xi-\xi'|}\,d\xi' - \int_{\xi}^{0^-} {}_1\bar{h}_0^4(\xi')\,e^{-|\xi-\xi'|}\,d\xi' - \int_{0^+}^{+\infty} {}_3\bar{h}_0^4(\xi')\,e^{-|\xi-\xi'|}\,d\xi'\right\} \quad (8b)$$

$${}_3v_0 + \left(\frac{\gamma - 1}{\gamma}\right)\frac{{}_3\bar{h}_0}{{}_3v_0} = 1 \quad (9a)$$

$$\left({}_3\bar{h}_0 + \frac{1}{2}{}_3v_0^2\right) = C - Q\left\{\int_{-\infty}^{0^-} {}_1\bar{h}_0^4(\xi')\,e^{-|\xi-\xi'|}\,d\xi' + \int_{0^+}^{\xi} {}_3\bar{h}_0^4(\xi')e^{-|\xi-\xi'|}\,d\xi' - \int_{\xi}^{+\infty} {}_3\bar{h}_0^4(\xi')\,e^{-|\xi-\xi'|}\,d\xi'\right\} \quad (9b)$$

Equations (7a) and (7b) are the shock-structure equations of the lowest order. It is clear that they do not exhibit a coupled radiation effect. The radiation terms in Eq. (7b) come in only as the modification due to the presence of the radiation field in the outside flow. These integrals can be evaluated with the help of the existing solutions for ${}_1\bar{h}_0(\xi)$ and ${}_3\bar{h}_0(\xi)$ in regions I and III. Equations (8) and (9) are the lowest-order equations outside the shock. The solution of these equations, with proper matching, supplies the necessary boundary conditions to determine the structure of the thin shock. The matching conditions are

$$\lim_{\xi \to 0^{\mp}} \{{}_{1,3}\bar{h}_0(\xi), {}_{1,3}v_0(\xi)\} = \lim_{y \to \mp\infty} \{\bar{h}_0(y), v_0(y)\}$$

It can easily be verified that the form of the usual Rankine-Hugoniot shock conditions just across the thin shock surface is unchanged. These relationships are

$${}_1v_0 + \left(\frac{\gamma - 1}{\gamma}\right)\frac{{}_1\bar{h}_0}{{}_1v_0} = {}_3v_0 + \left(\frac{\gamma - 1}{\gamma}\right)\frac{{}_3\bar{h}_0}{{}_3v_0} = 1 \quad (10a)$$

$$\lim_{\xi\to 0^-}\{{}_1\bar{h}_0(\xi) + \tfrac{1}{2}{}_1v_0^2(\xi)\} = \lim_{\xi\to 0^+}\{{}_3\bar{h}_0(\xi) + \tfrac{1}{2}{}_3v_0^2(\xi)\} = C - QJ \quad (10b)$$

where J is the definite integral

$$J = \int_{-\infty}^{0^-} {}_1\bar{h}_0^4(\xi')\,e^{-|\xi'|}\,d\xi' - \int_{0^+}^{+\infty} {}_3\bar{h}_0^4(\xi')\,e^{-|\xi'|}\,d\xi' \quad (11)$$

For the solution of the outside flow, Eqs. (8) and (9) can be combined into one single integral equation of the form

$$ {}_{iv}v_0{}^2 - \left(\frac{2\gamma}{\gamma+1}\right) {}_{iv}v_0 + \frac{c_2}{c_1{}^2}\left[\frac{2(\gamma-1)}{(\gamma+1)}\right] = \left[\frac{4m(\gamma-1)\sigma c_1{}^6}{\Gamma(\gamma+1)nR^4}\right]\int_{-\infty}^{+\infty}\left[{}_{iv}v_0(\xi') - {}_{iv}v_0{}^2(\xi')\right]^4 \operatorname{sgn}(\xi-\xi') \times e^{-|\xi-\xi'|}\,d\xi' \quad (12) $$

The subscript iv now stands for the distribution outside the shock. This equation had been subjected to thorough investigations in Refs. 1 and 2. Its solution supplies the boundary conditions at $\xi = 0^-$ and $\xi = 0^+$ and can be used to evaluate the definite integral J in the shock-structure problem. With this in mind, the shock-structure solution of Eqs. (7) can be expressed in either one of the two forms

$$ y - y_0{}^- = \int_{{}_{iv}v_0(0^-)}^{v_0} \frac{v_0 dv_0}{(\gamma+1)/2\gamma\, v_0{}^2 - v_0 + (\gamma-1)/2\gamma\,(C - QJ)} \quad (13) $$

$$ y\left(\frac{\gamma+1}{2\gamma}\right) = \left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{L\Gamma}{\mu}\right) \times = \left[\frac{{}_{iv}v_0(0^-)}{{}_{iv}v_0(0^-) - {}_{iv}v_0(0^+)}\right]\ln[{}_{iv}v_0(0^-) - V_0] - \left[\frac{{}_{iv}v_0(0^+)}{{}_{iv}v_0(0^-) - {}_{iv}v_0(0^+)}\right]\ln[v_0 - {}_{iv}v_0(0^+)] \quad (14) $$

Equation (14) bears essentially the same form as the classical Becker's solution for Prandtl number = 1 and μ = const.[7] However, the shape of the distribution will be different because ${}_{iv}v_0(0^+)$ and ${}_{iv}v_0(0^-)$ are different from the corresponding values across a radiationless shock, with the same upstream conditions. From Eq. (12), we observe that the condition of having a discontinuity in ${}_{iv}v_0$ across the shock is to require Eq. (18) to have real roots at $\xi = 0^-$ and 0^+. This condition is

$$ \left[\frac{\gamma^2}{2(\gamma^2-1)}\right] \geq \left[C - J\,\frac{2n\sigma c_1{}^6}{3\Gamma c_p{}^4}\right] $$

where J is defined in Eq. (11). Since J is never positive for a real shock, an important effect of thermal radiation is to reduce the velocity jump across the shock, and to increase the upstream Mach number necessary to produce a shock. These findings are in agreement with those of Heaslett and Baldwin[2] and show how the behavior across the discontinuity establishes the proper boundary conditions for their analysis. Results of Eq. (14) are plotted for several cases in Fig. 1.

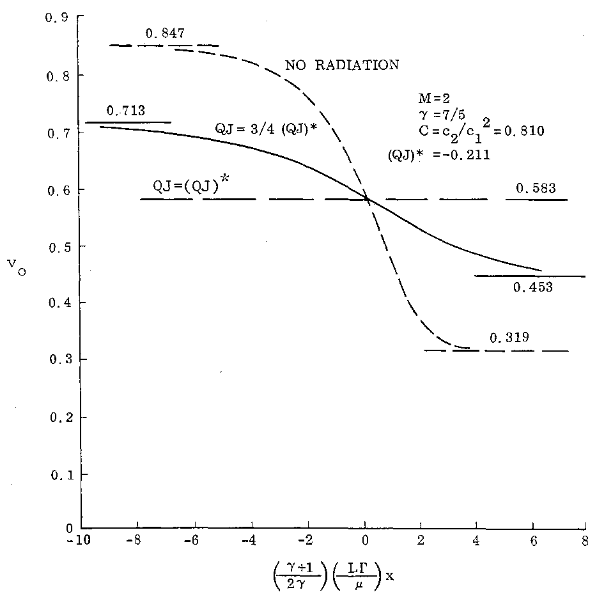

Fig. 1 Radiation effect on first-order structure of a typical shock.

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Linear Jet and Wake Solutions with Pressure Gradients

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Recently, Steiger and Bloom[1] presented the linear similar solutions for laminar jet and wake flows with pressure gradients. In the present paper the motivation is to establish a parallel and concise record of the similar solutions for turbulent flows. These solutions represent a useful and simple entré into the understanding of the more general flows with arbitrary pressure gradients.

In order to provide a convenient comparison, it was considered worthwhile to review the laminar flows and, in the process, some new considerations have been added to the work of Ref. 1. For our present purpose, we wish simplicity and so the development will be restricted to incompressible flow. The basic equations are

$$ (\partial/\partial x)(y^r u) + (\partial/\partial y)(y^r v) = 0 \quad (1 $$

$$ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = UU' + \frac{\nu_e}{y^r}\frac{\partial}{\partial y}\left(y^r\frac{\partial y}{\partial y}\right) \quad (2 $$

where $r = 0$ or 1 for either plane flow or axisymmetric flow ν_e is the "effective" viscosity and is either the molecule viscosity in the case of laminar flow or the eddy viscosity in th case of turbulent flow. The boundary conditions are tha $v(0) = \partial u(0)/\partial y = 0$ and $u \to U$ exponentially as $y \to \infty$ We now seek similar solutions of the form

$$ u(x, y) = U(x) + u_c(x)f(\eta) \qquad \eta = y/b(x) \quad (3 $$

where U is the mainstream velocity and $u_c(x) = u(x, 0) - U(x)$ is the centerline difference velocity. If (3) is inserte

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